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3 STABILITY OF SOLUTIONS TO PARTIAL DIFFERENTIAL
EQUATIONS 4

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I. Personnel

The technical personnel currently being partially supported by the research grant are as follows:

1. William G. Vogt, Principal Investigator - 1/2 time
2. Charles G. Krueger, Scientific Investigator - 1/4 time
3. Gabe R. Buis, Graduate Student - Full time

II. Research Accomplished

A. Bibliographical Survey.

A report [1] by Buis and Vogt, "The Stability Theory of Solutions to partial Differential Equations: A Bibliographical Survey" has been completed and copies have been forwarded to NASA Headquarters. The main conclusion which can be drawn from the survey [1] is that the stability theory of partial differential equations is, at best, just getting started. The contributions in this area are not mathematically rigorous and there is little or no justification for many of the results claimed.

B. Translations

Two translations [2,3] of significance to this research were completed by Buis as part of his Ph.D. research and forwarded to NASA Headquarters as reports.

C. Some Stability Problems in Hydrodynamics /

Buis and Vogt have completed a report [4] on some stability problems in hydrodynamics which has been forwarded to NASA Headquarters. The report concerns the physical systems which can be represented by:

$$\frac{\partial \underline{u}(t, \underline{x})}{\partial t} = \underline{L} \underline{u}(t, \underline{x}) \quad (1)$$

where $\underline{u}(t, \underline{x})$ is an n -vector function and \underline{L} is a matrix whose elements are linear or nonlinear differential operators specified on a bounded connected open subset Ω of an m -dimensional Euclidean space, E^m . The parameters of \underline{L} can be space and time dependent. In order to uniquely specify solutions to (1) a set of additional constraints or boundary conditions must be given by

$$\underline{H} \underline{u}(t, \underline{x}') = 0 \quad \underline{x}' \in \partial\Omega \quad (2)$$

where \underline{H} is a matrix whose elements are specified differential operators and $\partial\Omega$ is the boundary of Ω , $\bar{\Omega} = \Omega + \partial\Omega$.

Any solution of (1) and (2) will depend on some initial function $\underline{u}_0(\underline{x})$ belonging to the n -dimensional space of functions Θ , which in general will be a normed linear space. A solution starting at time t_0 and initial condition $\underline{u}_0(\underline{x}) \in \Theta$ will be indicated as $\underline{u}(t, \underline{x}; \underline{u}_0(\underline{x}), t_0)$. We will be interested in the stability properties of the equilibrium solutions, the solutions such that $\frac{\partial \underline{u}_{eq}(t, \underline{x})}{\partial t} = 0$ for all $t \geq t_0$, i.e. $\underline{u}(t, \underline{x}; \underline{u}_{eq}(\underline{x}), t_0) = \underline{u}_{eq}(\underline{x})$ for all $t \geq t_0$.

The stability of $\underline{u}_{eq}(\underline{x})$ is defined in terms of the norm, $||\cdot||$, of the normed linear space Θ by:

Definition. The equilibrium solution, $\underline{u}_{eq}(\underline{x})$, of (1) and (2) is said to be stable in the sense of Lyapunov if for every real number $\varepsilon > 0$, there exists a real number $\delta > 0$ such that for $\underline{u}_0(\underline{x}) \in \Theta$, $||\underline{u}_0(\underline{x}) - \underline{u}_{eq}(\underline{x})|| < \delta$ implies $||\underline{u}(t, \underline{x}; \underline{u}_0(\underline{x}), t_0) - \underline{u}_{eq}(\underline{x})|| < \varepsilon$ for all $t \geq t_0$.

Definition. $\underline{u}_{eq}(\underline{x})$ of (1) and (2) is said to be asymptotically stable if it is stable and in addition $||\underline{u}(t, \underline{x}; \underline{u}_0(\underline{x}), t_0) - \underline{u}_{eq}(\underline{x})|| \rightarrow 0$ as $t \rightarrow \infty$.

The main problem becomes then: what are the conditions on \underline{L} for the equilibrium solutions of (1) and (2) to be (asymptotically) stable? A principal factor is the selection of the space Θ , or rather, the selection of the norm which makes an abstract set of elements into a normed linear space. As distinct from Euclidean space, these norms are no longer all equivalent.

In [4] the emphasis is on the Hilbert spaces where the norm is defined by the inner product $\langle \cdot, \cdot \rangle$. Thus

$$||\underline{u}||^2 = \langle \underline{u}, \underline{u} \rangle \quad \text{for } \underline{u} \in \Theta.$$

The \underline{L} considered in [4] were linear and self-adjoint, $\langle \underline{L} \underline{v}, \underline{u} \rangle = \langle \underline{v}, \underline{L} \underline{u} \rangle$ for $\underline{u}, \underline{v} \in \Theta$. However rather than taking as inner product

$$\langle \underline{v}, \underline{u} \rangle = \int_{\Omega} \underline{v}^T \underline{u} \, d\Omega$$

where T is the transpose, a more general inner product is introduced by

$$\langle \underline{v}, \underline{u} \rangle = \int_{\Omega} \underline{v}^T \underline{W}(\underline{x}, t) \underline{u} \, d\Omega$$

where $\underline{W}(\underline{x}, t)$ is a weighting matrix with elements that can be chosen as continuous functions in both \underline{x} and t and continuously differentiable in t such that

$$\underline{W}(\underline{x}, t) = \underline{W}^T(\underline{x}, t)$$

and

$$\beta_1 \underline{u}^T \underline{u} \geq \underline{u}^T W(\underline{x}, t) \underline{u} \geq \beta_2 \underline{u}^T \underline{u}$$

$\infty > \beta_1 > \beta_2 > 0$ for all $\underline{x} \in \bar{\Omega}$, $t \in [t_0, \infty)$.

This choice of inner product enables us to expand the class of operators which are self-adjoint. However there are other practical implications too. Among these:

1. Rather than calculating the eigenvalues of \underline{L} it is often easier to apply integral inequalities. As such a considerable improvement of the parameter range results by making \underline{L} self-adjoint, i.e. introducing a general inner product where possible and applying these inequalities.
2. As shown in [4] stability conditions can be obtained for the equilibrium solutions of equations with nonlinear \underline{L} by means of imbedding in Θ . By using the general inner product an increase in maximally allowable perturbations can in many cases be achieved.
3. It is possible to eliminate the highest order odd derivative in linear differential operators \underline{L} which cannot be made self-adjoint to improve the range of parameter values for which the equilibrium solution is assured to be stable.

III. Research In Progress

From the relations discovered in [4], a natural extension leads to the consideration of semi groups of operators and their infinitesimal generators defined in a Banach Space or Hilbert Space. Within this framework it is possible to develop a stability theory for partial differential equations which closely parallels that for ordinary differential equations. A report is presently being prepared on this aspect of the theory.

IV. Future Directions For Research

From the developments described in III above it appears possible to extend the theory to nonlinear partial differential equations through the consideration of certain types of monotone operators in Banach spaces. The properties of monotone operators are presently being investigated.

References

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3. Ibrashev, Kh. I., "Construction of the Lyapunov Function for a Case of Stability with Constantly Acting Disturbances in Nonlinear Spaces", Akademiia Nauk Kazakhshoi SSR, Vestnik, Vol. 22, January, 1966, pp.42-46. Translated by G.R. Buis.
4. Buis, G. R. and W. G. Vogt, "Application of Lyapunov Stability Theory to Some Nonlinear Problems in Hydrodynamics ".